Electronic Journal of Qualitative Theory of Differential Equations
2005, No. **10**, 1-22; **http://www.math.u-szeged.hu/ejqtde/**

Nonlinear boundary value problem for nonlinear second order differential equations with impulses

Jan Tomeček

Abstract

The paper deals with the impulsive nonlinear boundary value problem

$$u''(t) = f(t, u(t), u'(t)) \quad \text{for a. e. } t \in [0, T],$$

$$u(t_j+) = J_j(u(t_j)), \quad u'(t_j+) = M_j(u'(t_j)), \quad j = 1, \ldots, m,$$

$$g_1(u(0), u(T)) = 0, \quad g_2(u'(0), u'(T)) = 0,$$

where $f \in Car([0,T] \times \mathbb{R}^2)$, g_1, $g_2 \in C(\mathbb{R}^2)$, J_j, $M_j \in C(\mathbb{R})$. An existence theorem is proved for non–ordered lower and upper functions. Proofs are based on the Leray–Schauder degree and on the method of a priori estimates.

Mathematics Subject Classification 2000: 34B37, 34B15.

Key words: Ordinary differential equation of the second order, well–ordered lower and upper functions, non–ordered lower and upper functions, nonlinear boundary value conditions, impulses.

1 Introduction

The nonlinear impulsive boundary value problem (IBVP) of the second order with nonlinear boundary conditions has been studied by many authors by the lower and upper functions method. For instance, the paper [1] considers such problem provided the nonlinearity in the equation satisfies the Nagumo growth conditions. In [2] the Nagumo conditions are replaced with other ones, which allow more than the quadratic growth of the right–hand side of the differential equation in the third variable. Both these works deal with well–ordered lower and upper functions. Until now there are no existence results available for the above problem such that its lower and upper functions are not well–ordered. The aim of this paper is to fill in this gap. The arguments are based on the ideas of papers [4] and [5], where the periodic nonlinear IBVP in the non–ordered case is investigated.

The paper is organized as follows. The first section contains basic notation and definitions. In the second section the Leray–Schauder Degree Theorem is established (Theorem 9) for the well–ordered case, which is used to prove the main existence result in the third section. As a secondary result the existence theorem with Nagumo conditions is obtained (Theorem 8). The third section contains the existence result (Theorem 21) for non–ordered lower and upper functions, where a Lebesgue bounded right–hand side of the differential equation is considered.

Let T be a positive real number. For a real valued function u defined a. e. on $[0,T]$, we put

$$\|u\|_\infty = \sup_{t\in[0,T]} \text{ess}\, |u(t)| \quad \text{ and } \quad \|u\|_1 = \int_0^T |u(s)|\, \mathrm{d}s.$$

For $k \in \mathbb{N}$ and a given set $B \subset \mathbb{R}^k$, let $C(B)$ denote the set of real valued functions which are continuous on B. Furthermore, let $C^1([0,T])$ be the set of functions having continuous first derivative on $[0,T]$ and $L([0,T])$ the set of functions which are Lebesgue integrable on $[0,T]$.
Let $m \in \mathbb{N}$ and let

$$0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = T$$

be a division of the interval $[0,T]$. We denote $D = \{t_1, \ldots, t_m\}$ and define C_D (C^1_D) as the set of functions $u : [0,T] \to \mathbb{R}$ such that the function $u_{|(t_i,t_{i+1})}$ (and its derivative) is continuous and continuously extendable to $[t_i, t_{i+1}]$ for $i = 0, \ldots, m$, $u(t_i) = \lim_{t\to t_i-} u(t)$, $i = 1, \ldots, m+1$ and $u(0) = \lim_{t\to 0+} u(t)$. Moreover, AC_D (or AC^1_D) stands for the set of functions $u \in C_D$ (or $u \in C^1_D$) which are absolutely continuous (or have absolutely continuous first derivatives) on each subinterval (t_i, t_{i+1}), $i = 0, \ldots, m$. For $u \in C^1_D$ and $i = 1, \ldots, m+1$ we write

$$u'(t_i) = u'(t_i-) = \lim_{t\to t_i-} u'(t), \quad u'(0+) = \lim_{t\to 0+} u'(t) \tag{1}$$

and

$$\|u\|_D = \|u\|_\infty + \|u'\|_\infty.$$

Note that the set C^1_D becomes a Banach space when equipped with the norm $\|\cdot\|_D$ and with the usual algebraic operations. By the symbol $\mathbb{R}^+$ we denote the set of positive real numbers and $\mathbb{R}^+_0 = \mathbb{R}^+ \cup \{0\}$.

Let $k \in \mathbb{N}$. We say that $f : [0,T] \times S \to \mathbb{R}$, $S \subset \mathbb{R}^k$ satisfies the *Carathéodory conditions* on $[0,T] \times S$ if f has the following properties: (i) for each $x \in S$ the function $f(\cdot, x)$ is measurable on $[0,T]$; (ii) for almost each $t \in [0,T]$ the function $f(t, \cdot)$ is continuous on S; (iii) for each compact set $K \subset S$ there exists a function $m_K(t) \in L([0,T])$ such that $|f(t,x)| \le m_K(t)$ for a. e. $t \in [0,T]$ and all $x \in K$. For the set of functions satisfying the *Carathéodory conditions* on $[0,T] \times S$ we write $Car([0,T] \times S)$. For a subset Ω of a Banach space, $\mathrm{cl}(\Omega)$ stands for the closure of Ω, $\partial\Omega$ stands for the boundary of Ω.

We study the following boundary value problem with nonlinear boundary value conditions and impulses:

$$u''(t) = f(t, u(t), u'(t)), \tag{2}$$

$$u(t_i+) = J_i(u(t_i)), \quad u'(t_i+) = M_i(u'(t_i)), \quad i = 1, \ldots, m, \tag{3}$$

$$g_1(u(0), u(T)) = 0, \quad g_2(u'(0), u'(T)) = 0, \tag{4}$$

where $f \in Car([0,T] \times \mathbb{R}^2)$, $g_1, g_2 \in C(\mathbb{R}^2)$, $J_i, M_i \in C(\mathbb{R})$ and $u'(t_i)$ are understood in the sense of (1) for $i = 1, \ldots, m$.

Definition 1 A function $u \in AC^1_D$ which satisfies equation (2) for a. e. $t \in [0,T]$ and fulfils conditions (3) and (4) is called a solution to the problem (2) – (4).

Definition 2 A function $\sigma_k \in AC^1_D$ is called a lower (upper) function of the problem (2) – (4) provided the conditions

$$[\sigma_k''(t) - f(t, \sigma_k(t), \sigma_k'(t))](-1)^k \le 0 \quad \text{for a. e. } t \in [0,T], \tag{5}$$

$$\sigma_k(t_i+) = J_i(\sigma_k(t_i)), \quad [\sigma_k'(t_i+) - M_i(\sigma_k'(t_i))](-1)^k \le 0, \quad i = 1, \ldots, m, \tag{6}$$

$$g_1(\sigma_k(0), \sigma_k(T)) = 0, \quad g_2(\sigma_k'(0), \sigma_k'(T))(-1)^k \le 0, \tag{7}$$

where $k = 1$ ($k = 2$), are satisfied.

2 Well–ordered lower and upper functions

Throughout this section we assume:

$$\left.\begin{array}{l} \sigma_1 \text{ and } \sigma_2 \text{ are lower and upper functions, respectively,} \\ \text{of the problem (2) – (4) and } \sigma_1(t) \le \sigma_2(t) \text{ for } t \in [0,T], \end{array}\right\} \tag{8}$$

$$\sigma_1(t_i) \le x \le \sigma_2(t_i) \implies J_i(\sigma_1(t_i)) \le J_i(x) \le J_i(\sigma_2(t_i)), \tag{9}$$

$$\left.\begin{array}{lcl} y \le \sigma_1'(t_i) & \implies & M_i(y) \le M_i(\sigma_1'(t_i)), \\ y \ge \sigma_2'(t_i) & \implies & M_i(y) \ge M_i(\sigma_2'(t_i)), \end{array}\right\} \tag{10}$$

for $i = 1, \ldots, m$,

$$\left.\begin{array}{lll} x > \sigma_1(0) & \Longrightarrow & g_1(\sigma_1(0), \sigma_1(T)) \neq g_1(x, \sigma_1(T)), \\ x < \sigma_2(0) & \Longrightarrow & g_1(\sigma_2(0), \sigma_2(T)) \neq g_1(x, \sigma_2(T)), \end{array}\right\} \tag{11}$$

$$\left.\begin{array}{lll} \sigma_1(T) \leq y & \Longrightarrow & g_1(\sigma_1(0), \sigma_1(T)) \leq g_1(\sigma_1(0), y), \\ \sigma_2(T) \geq y & \Longrightarrow & g_1(\sigma_2(0), \sigma_2(T)) \geq g_1(\sigma_2(0), y), \end{array}\right\} \tag{12}$$

$$\left.\begin{array}{lll} x \geq \sigma_1'(0) \text{ and } y \leq \sigma_1'(T) & \Longrightarrow & g_2(\sigma_1'(0), \sigma_1'(T)) \leq g_2(x, y), \\ x \leq \sigma_2'(0) \text{ and } y \geq \sigma_2'(T) & \Longrightarrow & g_2(\sigma_2'(0), \sigma_2'(T)) \geq g_2(x, y). \end{array}\right\} \tag{13}$$

Remark 3 If we put

$$g_1(x, y) = y - x, \quad g_2(x, y) = x - y \tag{14}$$

for x, $y \in \mathbb{R}$, then (4) reduces to the periodic conditions

$$u(0) = u(T), \quad u'(0) = u'(T). \tag{15}$$

From (14) we see that g_1 is one-to-one in x, which implies that g_1 satisfies (11). Moreover, g_1 fulfils (12) because g_1 is increasing in y. Similarly, since g_2 is increasing in x and decreasing in y, we have that g_2 satisfies (13).

We consider functions $\tilde{f} \in Car([0, T] \times \mathbb{R}^2)$, $\tilde{J}_i$, $\tilde{M}_i \in C(\mathbb{R})$ for $i = 1, \ldots, m$ having the following properties:

$$\left.\begin{array}{l} \tilde{f}(t, x, y) < f(t, \sigma_1(t), \sigma_1'(t)) \quad \text{for} \quad \text{a. e. } t \in [0, T], \\ x < \sigma_1(t), \quad |y - \sigma_1'(t)| \leq \frac{\sigma_1(t) - x}{\sigma_1(t) - x + 1}, \\ \tilde{f}(t, x, y) > f(t, \sigma_2(t), \sigma_2'(t)) \quad \text{for} \quad \text{a. e. } t \in [0, T], \\ x > \sigma_2(t), \quad |y - \sigma_2'(t)| \leq \frac{x - \sigma_2(t)}{x - \sigma_2(t) + 1}, \end{array}\right\} \tag{16}$$

$$\left.\begin{array}{rll} x < \sigma_1(t_i) & \Longrightarrow & \tilde{J}_i(x) < J_i(\sigma_1(t_i)), \\ \sigma_1(t_i) \leq x \leq \sigma_2(t_i) & \Longrightarrow & \tilde{J}_i(x) = J_i(x), \\ x > \sigma_2(t_i) & \Longrightarrow & \tilde{J}_i(x) > J_i(\sigma_2(t_i)), \end{array}\right\} \tag{17}$$

$$\left.\begin{array}{lll} y \leq \sigma_1'(t_i) & \Longrightarrow & \tilde{M}_i(y) \leq M_i(\sigma_1'(t_i)), \\ y \geq \sigma_2'(t_i) & \Longrightarrow & \tilde{M}_i(y) \geq M_i(\sigma_2'(t_i)). \end{array}\right\} \tag{18}$$

Next, we consider d_0, $d_T \in \mathbb{R}$ such that

$$\sigma_1(0) \leq d_0 \leq \sigma_2(0), \qquad \sigma_1(T) \leq d_T \leq \sigma_2(T). \tag{19}$$

We define an auxiliary impulsive boundary value problem

$$u''(t) = \tilde{f}(t, u(t), u'(t)), \tag{20}$$

$$u(t_i+) = \tilde{J}_i(u(t_i)), \quad u'(t_i+) = \tilde{M}_i(u'(t_i)), \quad i = 1, \ldots, m, \tag{21}$$

$$u(0) = d_0, \quad u(T) = d_T, \tag{22}$$

where $f \in Car(J \times \mathbb{R}^2)$, $\tilde{J}_i$, $\tilde{M}_i \in C(\mathbb{R})$, $i = 1, \ldots, m$, d_0, $d_T \in \mathbb{R}$.

Definition 4 A function $u \in AC^1_D$ which satisfies equation (20) for a. e. $t \in [0,T]$ and fulfils conditions (21) and (22) is called a solution to the problem (20) – (22).

Lemma 5 *Let* (8) – (10), (16) – (19) *be true. Then a solution u to the problem* (20) – (22) *satisfies the inequalities*

$$\sigma_1 \le u \le \sigma_2 \qquad \textit{on} \;\; [0,T]. \tag{23}$$

Proof. Let u be a solution to the problem (20) – (22). Put $v(t) = u(t) - \sigma_2(t)$ for $t \in [0,T]$. Then, by (22), we have $v(0) \le 0$ and $v(T) \le 0$. The rest of the proof is exactly the same as the proof of Lemma 2.1 in [3]. □

Proposition 6 *Let* (8) – (13) *be true and let there exist $h \in L([0,T])$ such that*

$$|f(t,x,y)| \le h(t) \qquad \textit{for a. e.} \quad t \in [0,T] \textit{ and all } (x,y) \in [\sigma_1(t), \sigma_2(t)] \times \mathbb{R}. \tag{24}$$

Then there exists a solution u to the problem (2) – (4) *satisfying* (23).

Proof.
Step 1. We define

$$\triangle = \min_{i=0,\dots,m} (t_{i+1} - t_i), \tag{25}$$

$$c = \|h\|_1 + \frac{\|\sigma_1\|_\infty + \|\sigma_2\|_\infty}{\triangle} + \|\sigma_1'\|_\infty + \|\sigma_2'\|_\infty + 1, \tag{26}$$

$$\alpha(t,x) = \begin{cases} \sigma_1(t) & \text{for} \quad x < \sigma_1(t), \\ x & \text{for} \quad \sigma_1(t) \le x \le \sigma_2(t), \\ \sigma_2(t) & \text{for} \quad \sigma_2(t) < x, \end{cases} \tag{27}$$

for all $t \in [0,T]$, $x \in \mathbb{R}$,

$$\beta(y) = \begin{cases} y & \text{for} \quad |y| \le c, \\ c \operatorname{sgn} y & \text{for} \quad |y| > c, \end{cases}$$

$$\omega_i(t,\epsilon) = \sup\{|f(t,\sigma_i(t),\sigma_i'(t)) - f(t,\sigma_i(t),y)| : |\sigma_i'(t) - y| \le \epsilon\}$$

for a. e. $t \in [0,T]$ and for $\epsilon \in [0,1]$, $i = 1,2$,

$$\left.\begin{array}{c} \tilde{J}_i(x) = x + J_i(\alpha(t_i,x)) - \alpha(t_i,x), \\ \tilde{M}_i(y) = y + M_i(\beta(y)) - \beta(y), \end{array}\right\} \tag{28}$$

and

$$\tilde{f}(t,x,y) = \begin{cases} f(t,\sigma_1(t),y) - \omega_1\Big(t, \frac{\sigma_1(t)-x}{\sigma_1(t)-x+1}\Big) - \frac{\sigma_1(t)-x}{\sigma_1(t)-x+1} & \text{for} \quad x < \sigma_1(t), \\ f(t,x,y) & \text{for} \quad \sigma_1(t) \le x \le \sigma_2(t), \\ f(t,\sigma_2(t),y) + \omega_2\Big(t, \frac{x-\sigma_2(t)}{x-\sigma_2(t)+1}\Big) + \frac{x-\sigma_2(t)}{x-\sigma_2(t)+1} & \text{for} \quad \sigma_2(t) < x \end{cases} \tag{29}$$

for a. e. $t \in [0,T]$ and all $x, y \in \mathbb{R}$. It follows from [2], Lemma 4 that $\tilde{f} \in Car([0,T] \times \mathbb{R}^2)$. We consider the problem (20), (21) and

$$\left.\begin{array}{rcl} u(0) & = & \alpha(0, u(0) + g_1(u(0), u(T))), \\ u(T) & = & \alpha(T, u(T) + g_2(u'(0), u'(T))). \end{array}\right\} \tag{30}$$

Step 2. We will prove solvability of the problem (20), (21), (30). We define a function $G : [0,T] \times [0,T] \to \mathbb{R}$ by

$$G(t,s) = \begin{cases} \frac{s(t-T)}{T} & \text{for} \quad 0 \le s < t \le T, \\ \frac{t(s-T)}{T} & \text{for} \quad 0 \le t \le s \le T, \end{cases} \tag{31}$$

and a totally continuous operator $\tilde{F} : C^1_D \to C^1_D$ by

$$\begin{aligned} (\tilde{F}u)(t) = {} & \frac{T-t}{T}\alpha(0, u(0) + g_1(u(0), u(T))) + \frac{t}{T}\alpha(T, u(T) + g_2(u'(0), u'(T))) \\ + {} & \int_0^T G(t,s)\tilde{f}(s, u(s), u'(s))\,\mathrm{d}s - \sum_{i=1}^m \frac{\partial G}{\partial s}(t, t_i)(\tilde{J}_i(u(t_i)) - u(t_i)) \\ + {} & \sum_{i=1}^m G(t, t_i)(\tilde{M}_i(u'(t_i)) - u'(t_i)), \end{aligned} \tag{32}$$

where $\frac{\partial G}{\partial s}(t,s)$ is continuous on $[0,T] \times [t,T]$. Obviously, u is a solution to the problem (20), (21), (30) if and only if u is a fixed point of the operator $\tilde{F}$.

We consider the family of equations

$$(I - \lambda\tilde{F})u = 0, \qquad \lambda \in [0,1]. \tag{33}$$

For $R > 0$ we define $B(R) = \{u \in C^1_D : \|u\|_D < R\}$. Relations (24), (27), (28), (29) imply that there exists $R_0 > 0$ such that $u \in B(R_0)$ for every solution u to the problem (33) and each $\lambda \in [0,1]$. Thus, $I - \lambda\tilde{F}$ is a homotopy on $\mathrm{cl}(B(R)) \times [0,1]$ for $R \ge R_0$, $\lambda \in [0,1]$ and

$$\deg(I - \tilde{\lambda}\tilde{F}, B(R)) = \deg(I - \lambda\tilde{F}, B(R))$$

for λ, $\tilde{\lambda} \in [0,1]$. Since $\deg(I, B(R)) = 1$, we conclude that

$$\deg(I - \tilde{F}, B(R)) = 1 \quad \text{for} \quad R \ge R_0. \tag{34}$$

Thus there exists a fixed point of $\tilde{F}$ in $B(R)$ and the problem (20), (21), (30) is solvable.

Step 3. Let u be a solution to (20), (21), (30). The definitions of the functions $\tilde{f}$, $\tilde{J}_i$, $\tilde{M}_i$ for $i = 1, \ldots, m$ and (10) imply that (16), (17), (18) are valid. We put $d_0 = \alpha(0, u(0) + g_1(u(0), u(T)))$ and $d_T = \alpha(T, u(T) + g_2(u'(0), u'(T)))$. Obviously, (19) is satisfied. We are allowed to use Lemma 5 and get (23). This fact together with (29) and (28) implies that u satisfies (2) and the first condition in (3). From the Mean Value Theorem it follows that for $i = 1, \dots, m$ there exists $\xi_i \in (t_i, t_{i+1})$ such that

$$|u'(\xi_i)| < \frac{1}{\triangle}(\|\sigma_1\|_\infty + \|\sigma_2\|_\infty).$$

Due to (24) and (26) we can see that $\|u'\|_\infty \leq c$, which together with (28) implies that u satisfies the second condition in (3).

Step 4. It remains to prove the validity of (4). It is sufficient to prove the inequalities

$$\sigma_1(0) \leq u(0) + g_1(u(0), u(T)) \leq \sigma_2(0) \tag{35}$$

and

$$\sigma_1(T) \leq u(T) + g_2(u'(0), u'(T)) \leq \sigma_2(T). \tag{36}$$

Let us suppose that the first inequality in (35) is not true. Then

$$\sigma_1(0) > u(0) + g_1(u(0), u(T)).$$

In view of (27) and (30) we have $u(0) = \sigma_1(0)$, thus it follows from (12) and (23) that

$$0 > g_1(\sigma_1(0), u(T)) \geq g_1(\sigma_1(0), \sigma_1(T)),$$

which contradicts (7). We prove the second inequality in (35) similarly. Let us suppose that the first inequality in (36) is not valid, i. e. let

$$\sigma_1(T) > u(T) + g_2(u'(0), u'(T)). \tag{37}$$

It follows from (27) and (30) that

$$u(T) = \sigma_1(T) \tag{38}$$

and by (37) we obtain that $0 > g_2(u'(0), u'(T))$. Further, by virtue of (7), (30), (35) and (38), we have

$$g_1(\sigma_1(0), \sigma_1(T)) = 0 = g_1(u(0), u(T)) = g_1(u(0), \sigma_1(T)).$$

In view of (23) and (11) we get

$$u(0) = \sigma_1(0). \tag{39}$$

It follows from (23), (38) and (39) that $\sigma_1'(T) \geq u'(T)$ and $u'(0) \geq \sigma_1'(0)$. Finally, by (13), we get the inequalities

$$0 > g_2(u'(0), u'(T)) \geq g_2(\sigma_1'(0), \sigma_1'(T)),$$

contrary to (7). The second inequality in (36) can be proved by a similar argument. Due to (30), the conditions (35) and (36) imply (4). $\square$

We can combine Proposition 6 with lemmas on a priori estimates to get the existence of solutions to the problem (2) – (4) when f does not fulfil (24). Here we will use the following lemma from the paper [3]. The existence result is contained in Theorem 8.

Lemma 7 *Assume that* $r > 0$ *and that*

$$k \in L([0,T]) \quad \textit{is nonnegative a. e. on } [0,T], \tag{40}$$

$$\omega \in C([1,\infty)) \quad \textit{is positive on } [1,\infty) \quad \textit{and} \quad \int_1^\infty \frac{\mathrm{d}s}{\omega(s)} = \infty. \tag{41}$$

Then there exists $r^* > 0$ *such that for each function* $u \in AC^1_D$ *satisfying* $\|u\|_\infty \le r$ *and*

$$|u''(t)| \le \omega(|u'(t)|)(|u'(t)| + k(t)) \tag{42}$$

for a. e. $t \in [0,T]$ *and for* $|u'(t)| > 1$, *the estimate*

$$\|u'\| \le r^* \tag{43}$$

holds.

Theorem 8 *Assume that* (8) – (13) *hold. Further, let*

$$|f(t,x,y)| \le \omega(|y|)(|y| + k(t)) \tag{44}$$

for a. e. $t \in [0,T]$ *and for each* $x \in [\sigma_1(t), \sigma_2(t)]$, $|y| > 1$, *where* k *and* ω *fulfil* (40) *and* (41). *Then the problem* (2) – (4) *has a solution* u *satisfying* (23).

Proof. It is formally the same as the proof of Theorem 3.1 in [3]. We use Proposition 6 instead of Proposition 3.2 in [3]. □

Now consider an operator $F : C^1_D \to C^1_D$ given by the formula

$$\begin{aligned}(Fu)(t) = {} & \frac{T-t}{T}(u(0) + g_1(u(0), u(T))) + \frac{t}{T}(u(T) + g_2(u'(0), u'(T))) \\ & + \int_0^T G(t,s) f(s, u(s), u'(s))\, \mathrm{d}s - \sum_{i=1}^m \frac{\partial G}{\partial s}(t, t_i)(J_i(u(t_i)) - u(t_i)) \\ & + \sum_{i=1}^m G(t, t_i)(M_i(u'(t_i)) - u'(t_i)).\end{aligned} \tag{45}$$

The main result of this section is the computation of the Leray–Schauder topological degree of the operator $I - F$ on a certain set Ω which is described by means of lower and upper functions σ_1, σ_2. The degree will be denoted by "deg". The degree computation will be used in the next section.

Theorem 9 *Let* $f \in Car([0,T] \times \mathbb{R}^2)$, g_1, $g_2 \in C(\mathbb{R}^2)$, J_i, $M_i \in C(\mathbb{R})$ *for* $i = 1, \dots, m$ *and let* σ_1, σ_2 *be lower and upper functions of the problem* (2) – (4) *such that*

$$\sigma_1 < \sigma_2 \quad \textit{on } [0,T] \quad \textit{and} \quad \sigma_1(t_i+) < \sigma_2(t_i+) \quad \textit{for} \quad i = 1, \dots, m,$$

$$\sigma_1(t_i) < x < \sigma_2(t_i) \implies J_i(\sigma_1(t_i)) < J_i(x) < J_i(\sigma_2(t_i)),$$
$$y \le \sigma_1'(t_i) \implies M_i(y) \le M_i(\sigma_1'(t_i)),$$
$$y \ge \sigma_2'(t_i) \implies M_i(y) \ge M_i(\sigma_2'(t_i))$$

for $i = 1, \dots, m$. *Let* (11) – (13) *be valid and let* $h \in L([0,T])$ *be such that*

$$|f(t,x,y)| \le h(t) \qquad \textit{for a. e. } t \in [0,T] \quad \textit{and all } (x,y) \in [\sigma_1(t), \sigma_2(t)] \times \mathbb{R}.$$

We define the (totally continuous) operator F *by* (45) *and* c *by* (26) *and denote*

$$\Omega = \{u \in C_D^1 : \|u'\| < c, \quad \sigma_1(t) < u(t) < \sigma_2(t) \quad \textit{for } t \in [0,T], \\ \sigma_1(t_i+) < u(t_i+) < \sigma_2(t_i+) \quad \textit{for } i = 1, \dots, m\}. \tag{46}$$

Then $\deg(I - F, \Omega) = 1$ *whenever* $Fu \ne u$ *on* $\partial\Omega$.

Proof. We consider $\tilde{J}_i$, $\tilde{M}_i$, $i = 1, \dots, m$ defined by (28) and $\tilde{f}$ by (29). Define $\tilde{F}$ by (32). We can see (use STEP 4 from the proof of Proposition 6) that

$$Fu = u \qquad \text{if and only if} \qquad \tilde{F}u = u \qquad \text{on } \mathrm{cl}(\Omega). \tag{47}$$

We suppose that $Fu \ne u$ for each $u \in \partial\Omega$. Then

$$\tilde{F}u \ne u \qquad \text{on } \partial\Omega.$$

It follows from STEP 3 of the proof of Proposition 6 that each fixed point u of $\tilde{F}$ satisfies (23) and consequently,

$$|u''(t)| = |f(t, u(t), u'(t))| \le h(t).$$

Then

$$\|u'\|_\infty \le \|h\|_1 + \frac{\|\sigma_1\|_\infty + \|\sigma_2\|_\infty}{\triangle} < c.$$

It means that

$$u = \tilde{F}u \implies u \in \Omega.$$

Now we choose $R > 0$ in (34) such that $\Omega \subset B(R)$. Let $\Omega_1 = \{u \in \Omega : (Fu)(0) \in [\sigma_1(0), \sigma_2(0)]\}$. If $u \in \Omega$ is a fixed point of F, then $u \in \Omega_1$. Hence F and (by (47)) $\tilde{F}$ have no fixed points in $\mathrm{cl}(\Omega) \setminus \Omega_1$. Moreover,

$$F = \tilde{F} \quad \text{on } \mathrm{cl}(\Omega_1).$$

Therefore, by the excision property,

$$\deg(I - F, \Omega) = \deg(I - F, \Omega_1) = \deg(I - \tilde{F}, \Omega_1) = \deg(I - \tilde{F}, B(R)) = 1.$$

This completes the proof. $\square$

3 Non–ordered lower and upper functions

We consider the following assumptions:

$$\left.\begin{array}{l}\sigma_1,\,\sigma_2 \text{ are lower and upper functions of the problem (2) – (4)},\\ \text{there exists}\,\tau\in[0,T]\,\text{such that}\,\sigma_1(\tau)>\sigma_2(\tau),\end{array}\right\}\tag{48}$$

$$\left.\begin{array}{lcl} x>\sigma_1(t_i) & \Longrightarrow & J_i(x)>J_i(\sigma_1(t_i)),\\ x<\sigma_2(t_i) & \Longrightarrow & J_i(x)<J_i(\sigma_2(t_i)),\end{array}\right\}\tag{49}$$

$$\left.\begin{array}{lcl} y\le\sigma_1'(t_i) & \Longrightarrow & M_i(y)\le M_i(\sigma_1'(t_i)),\\ y\ge\sigma_2'(t_i) & \Longrightarrow & M_i(y)\ge M_i(\sigma_2'(t_i))\end{array}\right\}\tag{50}$$

for $i=1,\dots,m$,

$$\left.\begin{array}{ll} g_1(x,y) & \text{is strictly decreasing in } x,\ \text{strictly increasing in } y,\\ g_2(x,y) & \text{is strictly increasing in } x,\ \text{strictly decreasing in } y,\end{array}\right\}\tag{51}$$

$$\lim_{x\to\pm\infty}|J_i(x)|=\infty,\qquad i=1,\dots,m,\tag{52}$$

$$\lim_{y\to\pm\infty}|M_i(y)|=\infty,\qquad i=1,\dots,m.\tag{53}$$

Remark 10 The assumptions (51) allow us to write (4) in the form

$$u(T)=h_1(u(0)),\quad u'(T)=h_2(u'(0)),$$

where $h_j:(a_j,b_j)\to\mathbb{R}$ is increasing, $-\infty\le a_j<b_j\le\infty$ for $j=1,2$. In this case, conditions (7) can be replaced by

$$\sigma_k(T)=h_1(\sigma_k(0)),\quad [h_2(\sigma_k'(0))-\sigma_k'(T)](-1)^k\le 0.$$

Definition 11 We define an operator $\mathcal{K}:C(\mathbb{R})\times\mathbb{R}^+\times\mathbb{R}_0^+\to C(\mathbb{R})$ by

$$\mathcal{K}(N,A,q)(x)=\begin{cases} x+q & \text{if } x\le -A-1,\\ N(-A)(A+1+x)-(x+q)(x+A) & \text{if } -A-1<x<-A,\\ N(x) & \text{if } -A\le x\le A,\\ N(A)(A+1-x)+(x-q)(x-A) & \text{if } A<x<A+1,\\ x-q & \text{if } x\ge A+1\end{cases}\tag{54}$$

for each $N\in C(\mathbb{R})$, $A>0$, $q\ge 0$. For the sake of simplicity of notation we will write $\tilde N(x;A,q)=\mathcal{K}(N,A,q)(x)$ for each $x\in\mathbb{R}$, $N\in C(\mathbb{R})$, $A>0$, $q\ge 0$.

Lemma 12 *Let $N\in C(\mathbb{R})$. Then the condition*

$$\forall K>0\ \exists L>0\ \forall x\in\mathbb{R}:\quad |N(x)|<K\Longrightarrow |x|<L\tag{55}$$

implies

$$\forall q\ge 0\ \forall K>0\ \exists \bar L>0\ \forall x\in\mathbb{R}\ \forall A>q:\quad |\tilde N(x;A,q)|<K\Longrightarrow |x|<\bar L;\tag{56}$$

thus, the constant $\bar L$ does not depend on A, but it does on q.

Proof. Let (55) be valid and $A > q \geq 0$. First we see that

$$\min\{N(A), A-q\} \leq \tilde{N}(x; A, q) \leq \max\{N(A), A-q+1\} \tag{57}$$

for $x \in [A, A+1]$ and an analogous assertion is valid for $x \in [-A-1, -A]$. Let $K > 0$ be arbitrary. We distinguish several cases. If $K \geq A-q+1$, then we can set $\bar{L} = K+q$. If $A-q \leq K < A-q+1$, then $\bar{L} = K+q+1$. Let us consider the case for which $K < A-q$. If $\min\{|N(A)|, |N(-A)|\} \geq K$, then in view of (57) and (55) we take $\bar{L} = L$, where L is the constant from (55). If $\max\{|N(A)|, |N(-A)|\} \leq K$, $\min\{|N(A)|, |N(-A)|\} \leq K$ or $\max\{|N(A)|, |N(-A)|\} \geq K$, we take $\bar{L} = L+1$. In general we can put $\bar{L} = K+L+q+1$. □

Lemma 13 *Let $N \in C(\mathbb{R})$, $A \geq q > 0$. Then*

$$\sup\{|N(x)| : |x| < a\} < b \quad \Longrightarrow \quad \sup\{|\tilde{N}(x; A, q)| : |x| < a\} < b$$

for $a > 0$, $b > a+q$.

Proof. Let us assume $a > 0$, $b > a+q$ and denote $N^* = \sup\{|N(x)| : |x| < a\}$. Then

$$\sup\{|\tilde{N}(x; A, q)| : |x| < a\} \leq \max\{a+q, N^*\} < b.$$

□

Lemma 14 *Let $\rho_1 > 0$, $\bar{h} \in L([0,T])$, $M_i \in C(\mathbb{R})$, $i = 1, \ldots, m$ satisfy* (53). *Then there exists $d > \rho_1$ such that the estimate*

$$\|u'\|_\infty < d \tag{58}$$

is valid for every $b > 0$ and every $u \in AC^1_D$ satisfying the conditions

$$|u'(\xi_u)| < \rho_1 \qquad \textit{for some} \quad \xi_u \in [0,T], \tag{59}$$

$$u'(t_i+) = \tilde{M}_i(u'(t_i); b, 0), \qquad i = 1, \ldots, m, \tag{60}$$

$$|u''(t)| < \bar{h}(t) \qquad \textit{for a. e. } t \in [0,T], \tag{61}$$

where $\tilde{M}_i(y; b, 0)$ is defined in the sense of Definition 11 for $i = 1, \ldots, m$.

Proof. Let us denote

$$b_i(a) = \sup_{|y|<a} |\tilde{M}_i(y; b, 0)| < \infty$$

for all $a > 0$, $i = 1, \ldots, m$. Let $\xi_u \in (t_j, t_{j+1}]$ for some $j \in \{0, \ldots, m\}$. Then in view of (59), (61) we get

$$|u'(t)| < \rho_1 + \|\bar{h}\|_1 = a_j \quad \text{for } t \in (t_j, t_{j+1}]. \tag{62}$$

Case A. If $j < m$, then due to (60) the inequalities $|u'(t_{j+1}+)| < b_{j+1}(a_j)$ and $|u'(t)| < b_{j+1}(a_j) + \|\bar{h}\|_1 = a_{j+1}$ are valid for $t \in (t_{j+1}, t_{j+2}]$. We can proceed in this way till $j = m$. We get

$$|u'(t)| < \max\{a_j, \dots, a_m\} + 1 \qquad \text{for } t \in (t_j, T].$$

Case B. If $j > 0$, then we will establish an estimate of $|u'|$ on $[0, t_j]$. If follows from (62) and (60) that

$$|\tilde{M}_j(u'(t_j); b, 0)| < a_j + 1.$$

The assumption (53) implies that for every $K > 0$ and for every $i = 1, \dots, m$ there exists $L > 0$ such that for $y \in \mathbb{R}$

$$|M_i(y)| < K \Longrightarrow |y| < L.$$

Due to Lemma 12, $\tilde{M}_i(y; b, 0)$ has the same property as M_i independently of b. Thus, there exists $c_{j-1} = c_{j-1}(a_j + 1) > 0$ such that

$$|u'(t_j)| < c_{j-1}$$

and c_{j-1} is independent of b. Then $|u'(t)| < c_{j-1} + \|\bar{h}\|_1 = a_{j-1}$ for $t \in (t_{j-1}, t_j]$. We proceed till $j = 1$.
If $\xi_u = 0$, we can proceed in the estimation in the same way as in Case A. We put $d = \max\{a_j : j = 0, \dots, m\} + 1$. □

Lemma 15 *Let ρ_0, d, $q > 0$ and $J_i \in C(\mathbb{R})$, $i = 1, \dots, m$ satisfy (52). Then there exists $c > \rho_0 + q$ such that the estimate*

$$\|u\|_\infty < c \tag{63}$$

is valid for every $a > q$ and every $u \in C^1_D$ satisfying conditions (58),

$$|u(\tau_u)| < \rho_0 \qquad \textit{for some} \quad \tau_u \in [0, T], \tag{64}$$

$$u(t_i+) = \tilde{J}_i(u(t_i); a, q), \qquad i = 1, \dots, m, \tag{65}$$

where $\tilde{J}_i(x; a, q)$ is defined in the sense of Definition 11 for $i = 1, \dots, m$.

Proof. We argue in the same way as in the proof of Lemma 14. □

Lemma 16 *Let J_i, $M_i \in C(\mathbb{R})$, $i = 1, \dots, m$, and (49), (50) be valid, let $q > 0$, σ_1, $\sigma_2 \in AC^1_D$ satisfy the equalities in (6), let a, $b \in \mathbb{R}$ be such that*

$$a > \|\sigma_1\|_\infty + \|\sigma_2\|_\infty + q + 1 \qquad \textit{and} \qquad b > \|\sigma_1'\|_\infty + \|\sigma_2'\|_\infty + \bar{\rho} + 1, \tag{66}$$

where

$$\bar{\rho} = \sum_{i=1}^{m} \left(|M_i(\sigma_1'(t_i))| + |M_i(\sigma_2'(t_i))|\right), \tag{67}$$

and let $\tilde{J}_i(x; a, q)$, $\tilde{M}_i(y; b, 0)$ be defined in the sense of Definition 11. Then the implications

$$\begin{cases} x > \sigma_1(t_i) & \Longrightarrow \quad \tilde{J}_i(x; a, q) > \tilde{J}_i(\sigma_1(t_i); a, q) = J_i(\sigma_1(t_i)), \\ x < \sigma_2(t_i) & \Longrightarrow \quad \tilde{J}_i(x; a, q) < \tilde{J}_i(\sigma_2(t_i); a, q) = J_i(\sigma_2(t_i)), \end{cases} \tag{68}$$

$$\begin{cases} y \le \sigma_1'(t_i) & \Longrightarrow \quad \tilde{M}_i(y; b, 0) \le M_i(\sigma_1'(t_i)), \\ y \ge \sigma_2'(t_i) & \Longrightarrow \quad \tilde{M}_i(y; b, 0) \ge M_i(\sigma_2'(t_i)), \end{cases} \tag{69}$$

are valid for $i = 1, \ldots, m$.

Proof. Obviously, (68) is valid for $|x| \le a$. Let $x > a$. Then

$$x > \max\{|\sigma_1(t_i)|, |\sigma_2(t_i)|\}$$

and it is sufficient to prove the first implication in (68). The fact that $|\sigma_1(t_i)| < a$ implies $\tilde{J}_i(\sigma_1(t_i); a, q) = J_i(\sigma_1(t_i))$. From the first inequality in (66) we get $x - q > a - q > \|\sigma_1\|_\infty$ and thus (49) and (6) yield

$$\begin{aligned} \tilde{J}_i(x; a, q) &= J_i(a)(a + 1 - x) + (x - q)(x - a) > \\ &J_i(\sigma_1(t_i))(a + 1 - x) + J_i(\sigma_1(t_i))(x - a) = J_i(\sigma_1(t_i)) \end{aligned}$$

for $x \in (a, a + 1)$. If $x \ge a + 1$, then

$$\tilde{J}_i(x; a, q) = x - q > \sigma_1(t_i+) = J_i(\sigma_1(t_i)).$$

Similarly, if $x < -a$, it is sufficient to prove the second implication in (68). The implications in (69) are obvious for $|y| \le b$. Otherwise, due to (66) and (67) we get

$$\tilde{M}_i(y; b, 0) = M_i(b)(b + 1 - y) + y(y - b) > M_i(\sigma_2'(t_i))$$

for $y \in (b, b + 1)$ and

$$\tilde{M}_i(y; b, 0) = y > M_i(\sigma_2'(t_i))$$

for $y \ge b + 1$. Analogously, (69) is valid for $y < -b$. $\square$

Definition 17 We define an operator $\mathcal{L} : C(\mathbb{R}^2) \times \mathbb{R}^+ \times (\mathbb{R} \setminus \{0\}) \to C(\mathbb{R}^2)$ by

$$\mathcal{L}(g, A, K)(x, y) = \begin{cases} g(x, y) & \text{if} \quad (x, y) \in [-A, A]^2, \\ K(y - x) & \text{if} \quad (x, y) \in \mathbb{R}^2 \setminus (-A - 1, A + 1)^2 \end{cases}$$

and so that $\mathcal{L}(g, A, K)(x, y)$ is a linear function of the variable y on rectangles $[-A, A] \times [A, A+1]$ and $[-A, A] \times [-A-1, -A]$, a linear function of the variable x on $[-A-1, -A] \times [-A, A]$ and $[A, A+1] \times [-A, A]$ and a bilinear function on squares $[A, A+1] \times [A, A+1]$, $[-A-1, -A] \times [A, A+1]$, $[-A-1, -A] \times [-A-1, -A]$ and $[A, A+1] \times [-A-1, -A]$ for each $g \in C(\mathbb{R}^2)$, $A > 0$ and $K \in \mathbb{R} \setminus \{0\}$.

Let us consider functions g_1, $g_2 \in C(\mathbb{R}^2)$ satisfying (51) and a, $b > 0$. We put

$$\tilde{g}_1(x, y; a) = \mathcal{L}(g_1, a, \max\{|g_1(x, y)| : (x, y) \in [-a, a]^2\} + 1)(x, y), \tag{70}$$

$$\tilde{g}_2(x, y; b) = \mathcal{L}(g_2, b, -\max\{|g_2(x, y)| : (x, y) \in [-b, b]^2\} - 1)(x, y) \tag{71}$$

for each $(x, y) \in \mathbb{R}^2$.

Remark 18 The functions $\tilde{g}_1(x, y; a)$ and $\tilde{g}_2(x, y; b)$ defined by (70) and (71) preserve the properties of g_1 and g_2, respectively, in (51).

For arbitrary a, $b > 0$, we consider the conditions

$$\tilde{g}_1(u(0), u(T); a) = 0, \qquad \tilde{g}_2(u'(0), u'(T); b) = 0 \tag{72}$$

and for $u \in C^1_D$

$$u(s_u) < \sigma_1(s_u) \quad \text{and} \quad u(t_u) > \sigma_2(t_u) \quad \text{for some } s_u, t_u \in [0, T], \tag{73}$$

$$u \geq \sigma_1 \text{ on } [0, T] \quad \text{and} \quad \inf_{t \in [0,T]} |u(t) - \sigma_1(t)| = 0, \tag{74}$$

$$u \leq \sigma_2 \text{ on } [0, T] \quad \text{and} \quad \inf_{t \in [0,T]} |u(t) - \sigma_2(t)| = 0. \tag{75}$$

Lemma 19 *Let σ_1, $\sigma_2 \in AC^1_D$ be lower and upper functions of the problem* (2) – (4)*, let J_i, $M_i \in C(\mathbb{R})$ satisfy* (49), (50)*, let g_1, $g_2 \in C(\mathbb{R}^2)$ satisfy* (51)*, let $q > 0$, a, $b \in \mathbb{R}$ satisfy* (66)*, where $\bar{\rho}$ is defined in* (67)*. We define $\tilde{J}_i(x; a, q)$ and $\tilde{M}_i(y; b, 0)$ in the sense of Definition 11, $\tilde{g}_1(x, y; a)$, $\tilde{g}_2(x, y; b)$ by* (70), (71)*, respectively and*

$$B = \{u \in C^1_D : u \textit{ satisfies } (72), (60), (65) \\ \textit{and one of the conditions } (73), (74), (75)\ \}. \tag{76}$$

Then each function $u \in B$ satisfies

$$\begin{cases} |u'(\xi_u)| < \rho_1 & \textit{for some } \xi_u \in [0, T], \textit{ where} \\ \rho_1 = \frac{2}{t_1}\big(\|\sigma_1\|_\infty + \|\sigma_2\|_\infty\big) + \|\sigma_1'\|_\infty + \|\sigma_2'\|_\infty + 1. \end{cases} \tag{77}$$

Proof. PART 1. Let $u \in B$ satisfy (73). We consider three cases.
CASE A. If $\min\{\sigma_1(t), \sigma_2(t)\} \leq u(t) \leq \max\{\sigma_1(t), \sigma_2(t)\}$ for each $t \in [0, T]$, then it follows from the Mean Value Theorem that there exists $\xi_u \in (0, t_1)$ such that

$$|u'(\xi_u)| \leq \frac{2}{t_1}(\|\sigma_1\|_\infty + \|\sigma_2\|_\infty).$$

CASE B. Assume that $u(s) > \sigma_1(s)$ for some $s \in [0, T]$. We denote $v = u - \sigma_1$ on $[0, T]$. According to (73) we have

$$\inf_{t \in [0,T]} v(t) < 0 \quad \text{and} \quad \sup_{t \in [0,T]} v(t) > 0. \tag{78}$$

We will prove that

$$v'(\alpha) = 0 \quad \text{for some } \alpha \in [0,T] \quad \text{or} \quad v'(\tau+) = 0 \quad \text{for some } \tau \in D. \tag{79}$$

Suppose, on the contrary, that (79) does not hold.
Let $v'(0) > 0$. In view of (66), (7), (72) and Remark 18 we have

$$\begin{aligned}\tilde{g}_2(\sigma_1'(0), \sigma_1'(T); b) &= g_2(\sigma_1'(0), \sigma_1'(T)) \geq 0 \\ &= \tilde{g}_2(u'(0), u'(T); b) > \tilde{g}_2(\sigma_1'(0), u'(T); b).\end{aligned}$$

Thus the monotony of $\tilde{g}_2$ yields $\sigma_1'(T) < u'(T)$, i. e. $v'(T) > 0$. Due to the fact that (79) does not hold and by virtue of (60) and (6) we get

$$0 < v'(t_m+) = u'(t_m+) - \sigma_1'(t_m+) \leq \tilde{M}_m(u'(t_m); b, 0) - M_m(\sigma_1'(t_m)).$$

In view of (69) we get $v'(t_m) > 0$. Continuing by induction we have

$$v'(t) > 0 \quad \text{for each } t \in [0,T] \quad \text{and} \quad v'(\tau+) > 0 \quad \text{for each } \tau \in D. \tag{80}$$

If $v(0) \geq 0$, then due to (80) we have $v > 0$ on $(0, t_1]$. The first implication in (68) implies $u(t_1+) > \sigma_1(t_1+)$. We proceed till $t = t_{m+1}$. We get $v \geq 0$ on $[0,T]$, which contradicts (78). If $v(0) < 0$ then in view of Remark 18, (7), (72) and (66) we get

$$\begin{aligned}\tilde{g}_1(u(0), \sigma_1(T); a) &> \tilde{g}_1(\sigma_1(0), \sigma_1(T); a) \\ &= g_1(\sigma_1(0), \sigma_1(T)) = 0 = \tilde{g}_1(u(0), u(T); a).\end{aligned}$$

Thus $v(T) < 0$. Due to (80) we have $v < 0$ on $(t_m, T]$ and the relations (65) and (6) imply $\tilde{J}_m(u(t_m); a, q) < J_m(\sigma_1(t_m))$. Due to (68) we get $u(t_m) \leq \sigma_1(t_m)$. We proceed in the same way till $t = t_0$. The inequality $v \leq 0$ on $[0,T]$ contradicts (78).
Let $v'(0) < 0$, then $v'(t_1) < 0$. In view of (60), (69) and (6) we have

$$u'(t_1+) = \tilde{M}_1(u'(t_1); b, 0) \leq M_1(\sigma_1'(t_1)) \leq \sigma_1'(t_1+)$$

and (79) implies $v'(t_1+) < 0$. We proceed till $t = t_{m+1}$ again and get

$$v'(t) < 0 \quad \text{for each } t \in [0,T] \quad \text{and} \quad v'(\tau+) < 0 \quad \text{for each } \tau \in D. \tag{81}$$

We distinguish two cases

$$v(0) \geq 0 \quad \text{and} \quad v(0) < 0.$$

In an analogous way we get a contradiction to (78). Assertion (79) implies that there exists $\xi_u \in [0,T]$ such that

$$|u'(\xi_u)| < \|\sigma_1'\|_\infty + 1.$$

CASE C. Assume that $u(s) < \sigma_2(s)$ for some $s \in [0,T]$. We can prove that there exists $\xi_u \in [0,T]$ such that

$$|u'(\xi_u)| < \|\sigma_2'\|_\infty + 1$$

by an argument analogous to CASE B.
PART 2. Assume that $u \in B$ satisfies (74). Then $u \geq \sigma_1$ on $[0,T]$ and either there exists $\alpha_u \in [0,T]$ such that $u(\alpha_u) = \sigma_1(\alpha_u)$ or there exists $t_j \in D$ such that $u(t_j+) = \sigma_1(t_j+)$.
CASE A. Assume that $\alpha_u \in (0,T) \setminus D$ is such that $u(\alpha_u) = \sigma_1(\alpha_u)$. Then $u'(\alpha_u) = \sigma_1'(\alpha_u)$ and (77) is valid. If $\alpha_u = 0$ then $u(0) = \sigma_1(0)$ and

$$\begin{aligned}\tilde{g}_1(u(0), \sigma_1(T); a) &= \tilde{g}_1(\sigma_1(0), \sigma_1(T); a)\\ &= g_1(\sigma_1(0), \sigma_1(T)) = 0 = \tilde{g}_1(u(0), u(T); a)\end{aligned}$$

according to (66), (7) and (72). The monotony of $\tilde{g}_1$ implies $\sigma_1(T) = u(T)$ and (74) gives

$$u'(0) \geq \sigma_1'(0) \quad \text{and} \quad u'(T) \leq \sigma_1'(T).$$

Due to (72), Remark 18, (66) and (7), we get

$$0 = \tilde{g}_2(u'(0), u'(T); b) \geq \tilde{g}_2(\sigma_1'(0), \sigma_1'(T); b) = g_2(\sigma_1'(0), \sigma_1'(T)) \geq 0,$$

thus $\sigma_1'(0) = u'(0)$ and $\sigma_1'(T) = u'(T)$, i. e. $\xi_u = 0$. If $\alpha_u = t_j \in D$ then $u(t_j) = \sigma_1(t_j)$,

$$u(t_j+) = \tilde{J}_j(u(t_j); a, q) = \tilde{J}_j(\sigma_1(t_j); a, q) = J_j(\sigma_1(t_j)) = \sigma_1(t_j+)$$

and from (74) we get the inequality $u'(t_j+) \geq \sigma_1'(t_j+)$ and $u'(t_j) \leq \sigma_1'(t_j)$. From (69) we have $\tilde{M}_j(u'(t_j); b, 0) \leq M_j(\sigma_1'(t_j))$, i. e. $u'(t_j+) \leq \sigma_1'(t_j+)$. We get $\sigma_1'(t_j+) = u'(t_j+)$. The estimate (77) is valid for ξ_u sufficiently close to t_j.
CASE B. If the second possibility occurs, i. e. $u(t_j+) = \sigma_1(t_j+)$ for some $t_j \in D$, then $\tilde{J}_j(u(t_j); a, q) = J_j(\sigma_1(t_j))$. We get $u(t_j) \leq \sigma_1(t_j)$ and due to (74) we have $u(t_j) = \sigma_1(t_j)$. Arguing as before, we get (77).
PART 3. Assume that $u \in B$ satisfies (75). We argue analogously to PART 2. □

We need the following lemma from the paper [4] (Lemma 2.4.), where the periodic problem was considered. The proof for our problem is formally the same.

Lemma 20 *Each $u \in B$ satisfies the condition*

$$\min\{\sigma_1(\tau_u+), \sigma_2(\tau_u+)\} \leq u(\tau_u+) \leq \max\{\sigma_1(\tau_u+), \sigma_2(\tau_u+)\} \tag{82}$$

for some $\tau_u \in [0,T)$.

Now, we are ready to prove the main result of this paper concerning the case of non–ordered lower and upper functions which is contained in the next theorem.

Theorem 21 *Assume that $f \in Car([0,T]\times\mathbb{R}^2)$, J_i, $M_i \in C(\mathbb{R})$ for $i = 1,\dots,m$, g_1, $g_2 \in C(\mathbb{R}^2)$, (48) – (53) hold and there exists $h \in L([0,T])$ such that*

$$|f(t,x,y)| \le h(t) \quad \textit{for a. e. } t \in [0,T] \textit{ and each } (x,y) \in \mathbb{R}^2. \tag{83}$$

Then the problem (2) – (4) has a solution u satisfying one of the conditions (73) – (75).

Proof.
Step 1. Let σ_1, σ_2 be lower and upper functions of (2) – (4) and let ρ_1 be defined by (77). We put $\bar{h} = 2h+1$ a. e. on $[0,T]$. By Lemma 14 we find $d > \rho_1$ satisfying (58). Provided $d > \rho_1 + \bar{\rho}$, where $\bar{\rho}$ is defined in (67), the properties of the constant d remain valid. We put $\rho_0 = \|\sigma_1\|_\infty + \|\sigma_2\|_\infty + 1$ and

$$q = \frac{T}{m}\sum_{i=1}^{m}\max\{\max_{|y|\le d+1}|M_i(y)|, d+1\}.$$

Lemma 15 guarantees the existence of $c > \rho_0 + q$ such that (63) is valid. Obviously,

$$c > \|\sigma_1\|_\infty + \|\sigma_2\|_\infty + q + 1 \quad \text{and} \quad d > \|\sigma_1'\|_\infty + \|\sigma_2'\|_\infty + \bar{\rho} + 1.$$

We define

$$\tilde{f}(t,x,y) = \begin{cases} f(t,x,y) - h(t) - 1 & \text{if } \ x \le -c-1, \\ f(t,x,y) + (x+c)(h(t)+1) & \text{if } \ -c-1 < x < -c, \\ f(t,x,y) & \text{if } \ -c \le x \le c, \\ f(t,x,y) + (x-c)(h(t)+1) & \text{if } \ c < x < c+1, \\ f(t,x,y) + h(t) + 1 & \text{if } \ x \ge c+1 \end{cases} \tag{84}$$

for a. e. $t \in [0,T]$ and each $(x,y) \in \mathbb{R}^2$,

$$\tilde{J}_i(x;c,q) = \mathcal{K}(J_i,c,q)(x), \quad i = 1,\dots,m, \tag{85}$$

$$\tilde{M}_i(y;d,0) = \mathcal{K}(M_i,d,0)(y), \quad i = 1,\dots,m, \tag{86}$$

$$\tilde{g}_1(x,y;c) = \mathcal{L}(g_1,c,\max\{|g_1(x,y)| : (x,y) \in [-c,c]^2\} + 1)(x,y), \tag{87}$$

$$\tilde{g}_2(x,y;d) = \mathcal{L}(g_2,d,-\max\{|g_2(x,y)| : (x,y) \in [-d,d]^2\} - 1)(x,y) \tag{88}$$

and consider the problem

$$u'' = \tilde{f}(t,u,u'), \tag{89}$$

$$u(t_i+) = \tilde{J}_i(u(t_i);c,q), \quad u'(t_i+) = \tilde{M}_i(u'(t_i);d,0), \quad i = 1,\dots,m, \tag{90}$$

$$\tilde{g}_1(u(0),u(T);c) = 0, \quad \tilde{g}_2(u'(0),u'(T);d) = 0. \tag{91}$$

It follows from definitions (84) – (88) that σ_1 and σ_2 are lower and upper functions of the problem (89) – (91). The inequality (83) implies

$$|\tilde{f}(t,x,y)| \le \bar{h}(t) \quad \text{for a. e. } t \in [0,T] \quad \text{and each } (x,y) \in \mathbb{R}^2 \tag{92}$$

and (84) yields

$$\begin{cases} \tilde{f}(t,x,y) < 0 & \text{for a. e. } t \in [0,T] \text{ and each } (x,y) \in (-\infty, -c-1] \times \mathbb{R}, \\ \tilde{f}(t,x,y) > 0 & \text{for a. e. } t \in [0,T] \text{ and each } (x,y) \in [c+1, \infty) \times \mathbb{R}. \end{cases} \tag{93}$$

STEP 2. We construct lower and upper functions σ_3, σ_4 of the problem (89) – (91). We put

$$A^* = q + \sum_{i=1}^{m} \max_{|x| \le c+1} |\tilde{J}_i(x; c, q)| \tag{94}$$

and

$$\begin{cases} \sigma_4(0) = A^* + mq, \\ \sigma_4(t) = A^* + (m-i)q + \frac{mq}{T} t, \quad t \in (t_i, t_{i+1}], \\ \sigma_3 = -\sigma_4 \quad \text{on } [0,T]. \end{cases} \tag{95}$$

It is obvious that σ_3, $\sigma_4 \in AC^1_D$ and

$$\begin{cases} \sigma_3(t) < -A^* < -c-1, \quad \sigma_4(t) > A^* > c+1, \\ \sigma_3'(t) = -\frac{mq}{T} \le -d-1, \quad \sigma_4'(t) = \frac{mq}{T} \ge d+1 \end{cases}$$

for $t \in [0,T]$. From these facts we can prove that σ_3 and σ_4 are lower and upper functions of the problem (89) – (91).

STEP 3. We define G by (31), the operator F by

$$\begin{aligned} (Fu)(t) &= \frac{T-t}{T}(u(0) + \tilde{g}_1(u(0), u(T); c)) + \frac{t}{T}(u(T) + \tilde{g}_2(u'(0), u'(T); d)) \\ &+ \int_0^T G(t,s)\tilde{f}(s, u(s), u'(s))\,\mathrm{d}s - \sum_{i=1}^{m} \frac{\partial G}{\partial s}(t, t_i)(\tilde{J}_i(u(t_i); c, q) - u(t_i)) \\ &+ \sum_{i=1}^{m} G(t, t_i)(\tilde{M}_i(u'(t_i); d, 0) - u'(t_i)) \end{aligned} \tag{96}$$

and its domain

$$\begin{aligned} \Omega_0 = \{u \in C^1_D : \|u'\|_\infty < C^*, \quad \sigma_3 < u < \sigma_4 \quad \text{on } [0,T], \\ \sigma_3(\tau+) < u(\tau+) < \sigma_4(\tau+), \quad \text{for } \tau \in D\} \end{aligned} \tag{97}$$

where

$$C^* = \|\bar{h}\|_1 + \frac{\|\sigma_3\|_\infty + \|\sigma_4\|_\infty}{\triangle} + \|\sigma_3'\|_\infty + \|\sigma_4'\|_\infty + 1 \tag{98}$$

($\triangle$ is defined in (25)). It is clear that u is a solution to the problem (89) – (91) if and only if $Fu = u$.
Step 4. We will prove that for every solution to (89) – (91) the implication

$$u \in \mathrm{cl}(\Omega_0) \Longrightarrow u \in \Omega_0$$

is true. Let

$$u \in \mathrm{cl}(\Omega_0) \tag{99}$$

be valid. For each $i = 1, \dots, m$ there exists $\xi_i \in (t_i, t_{i+1})$ such that

$$|u'(\xi_i)| \le \frac{\|\sigma_3\|_\infty + \|\sigma_4\|_\infty}{\triangle}.$$

Hence we get $\|u'\|_\infty < C^*$. It remains to show that $\sigma_3 < u < \sigma_4$ on $[0,T]$ and $\sigma_3(\tau+) < u(\tau+) < \sigma_4(\tau+)$ for $\tau \in D$.

Assume the contrary, i. e. let there exists $k \in \{3,4\}$ such that

$$u(\xi) = \sigma_k(\xi) \quad \text{for some } \xi \in [0,T] \tag{100}$$

or

$$u(\tau+) = \sigma_k(\tau+) \quad \text{for some } \tau \in D. \tag{101}$$

Case A. Let (100) be valid for $k = 4$.
(i) If $\xi = 0$, then in view of (91) we get $u(0) = \sigma_4(0) = \sigma_4(T) = u(T) = A^* + mq$ and $u'(0) = u'(T) = \frac{mq}{T} = \sigma_4'(t)$ for each $t \in [0,T]$. Then there exists $\delta > 0$ such that

$$u(t) > c + 1 \quad \text{for each } t \in [0,\delta]$$

and

$$u'(t) - u'(0) = \int_0^t \tilde{f}(s, u(s), u'(s))\,\mathrm{d}s > 0 \quad \text{for each } t \in [0,\delta].$$

We have $u'(t) > u'(0) = \sigma_4'(t)$ for every $t \in (0,\delta]$, thus $u > \sigma_4$ on $(0,\delta]$, which contradicts assumption (99).
(ii) If $\xi \in (t_i, t_{i+1})$ for some $i \in \{0, \dots, m\}$, then $u'(\xi) = \sigma_4'(\xi) = \frac{mq}{T} = \sigma_4'(t)$ for $t \in [0,T]$.
(iii) If $\xi = t_i \in D$ then $u(t_i) = \sigma_4(t_i)$ and

$$u(t_i+) = \sigma_4(t_i) - q > c + 1 - q > \|\sigma_1\|_\infty + \|\sigma_2\|_\infty.$$

From (99) we get $u'(t_i+) \le \sigma_4'(t_i+)$ and $u'(t_i) \ge \sigma_4'(t_i)$. This implies $u'(t_i+) \ge \sigma_4'(t_i+)$ and

$$u'(t_i+) = \sigma_4'(t_i+) = \frac{mq}{T} = \sigma_4'(t) \quad \text{for each } t \in [0,T].$$

We get a contradiction as in (i).
Case B. Let (101) be valid for $k = 4$, i. e. $u(t_i+) = \sigma_4(t_i+)$. Then

$$\tilde{J}_i(u(t_i); c, q) = \sigma_4(t_i+) = \sigma_4(t_i) - q > A^* - q$$

and (94) yields $u(t_i) > c+1$. We get $\tilde{J}_i(u(t_i); c, q) = u(t_i) - q$, thus $u(t_i) = \sigma_4(t_i)$. We get a contradiction as in (iii). We prove (100) and (101) for $k = 3$ analogously.
Step 5. We define

$$\Omega_1 = \{u \in \Omega_0 : u(t) > \sigma_1(t) \text{ for } t \in [0, T], \quad u(\tau+) > \sigma_1(\tau+) \text{ for } \tau \in D\},$$

$$\Omega_2 = \{u \in \Omega_0 : u(t) < \sigma_2(t) \text{ for } t \in [0, T], \quad u(\tau+) < \sigma_2(\tau+) \text{ for } \tau \in D\}$$

and

$$\tilde{\Omega} = \Omega_0 \setminus \mathrm{cl}(\Omega_1 \cup \Omega_2).$$

We see that

$$\tilde{\Omega} = \{u \in \Omega_0 : u \text{ satisfies } (73)\}.$$

Now, we will prove the implication

$$u \in \mathrm{cl}(\tilde{\Omega}) \Longrightarrow (\|u\|_\infty < c \quad \text{and} \quad \|u'\|_\infty < d) \tag{102}$$

for every solution u to the problem (89) – (91). Let u be a solution to the problem (89) – (91) and $u \in \mathrm{cl}(\tilde{\Omega})$, where

$$\mathrm{cl}(\tilde{\Omega}) = \{u \in \Omega_0 : u \text{ satisfies one of the conditions } (73), (74), (75)\}.$$

It means $u \in B$ (with constants c, d instead of a, b, respectively) and Lemma 19 implies that there exists $\xi_u \in [0, T]$ such that $|u'(\xi_u)| < \rho_1$. We apply Lemma 14 and get $\|u'\|_\infty < d$. From Lemma 20 and Lemma 15 we get (102).
Step 6. Finally, we will prove the existence result for the problem (2) – (4). We consider the operator F defined by (96).
Case A. Let F have a fixed point u on $\partial\tilde{\Omega}$, i. e. $u \in \partial\tilde{\Omega}$. Then (102) implies that u is a solution to the problem (2) – (4).
Case B. Let $Fu \neq u$ for each $u \in \partial\tilde{\Omega}$. Then $Fu \neq u$ for each $u \in \partial\Omega_0 \cup \partial\Omega_1 \cup \partial\Omega_2$. Theorem 9 implies

$$\begin{aligned}
\deg(I - F, \Omega(\sigma_3, \sigma_4, C^*)) &= \deg(I - F, \Omega_0) = 1,\\
\deg(I - F, \Omega(\sigma_1, \sigma_4, C^*)) &= \deg(I - F, \Omega_1) = 1,\\
\deg(I - F, \Omega(\sigma_3, \sigma_2, C^*)) &= \deg(I - F, \Omega_2) = 1.
\end{aligned}$$

Using the additivity property of the Leray–Schauder topological degree we obtain

$$\deg(I - F, \tilde{\Omega}) = \deg(I - F, \Omega_0) - \deg(I - F, \Omega_1) - \deg(I - F, \Omega_2) = -1.$$

Therefore F has a fixed point $u \in \tilde{\Omega}$. From (102) we conclude that $\|u\|_\infty < c$ and $\|u'\|_\infty < d$. From (84) – (91) we see that u is a solution to the problem (2) – (4). The proof is complete. □

Example 22 Let us consider the interval $[0, T] = [0, 2]$, one impulsive point $t_1 = 1$ and the boundary value problem

$$u''(t) = e^t - 10 \operatorname{arctg}(u(t) + 3u'(t)) \quad \text{for a. e. } t \in [0, 2],$$
$$u(1+) = k_1 u(1), \quad u'(1+) = k_2 u'(1), \quad k_1, k_2 \in (0, 1),\ k_1 \geq k_2,$$
$$u(0) = u(2), \quad u'^3(0) = u'(2).$$

We will seek a lower function of this problem in the form

$$\sigma_1(t) = \begin{cases} a + bt & \text{for} \quad t \in [0, 1], \\ c + dt & \text{for} \quad t \in (1, 2]. \end{cases} \tag{103}$$

First we choose $b > 1$. From the impulsive and boundary value conditions (6) and (7) respectively we get relations

$$c + d = k_1(a + b), \quad a = c + 2d, \quad k_2 b \leq d \leq b^3.$$

We can put

$$d = k_2 b, \quad a = \frac{k_1 + k_2}{1 - k_1} b, \quad c = \frac{k_1 + 2k_1k_2 - k_2}{1 - k_1} b.$$

Then a, b, c, $d > 0$ and all these constants become sufficiently large, when b is taken sufficiently large. The condition (5) can be satisfied for a suitable b. Analogously we seek an upper function of this problem (for example $\sigma_2 = -\sigma_1$ for suitable constants). We can see the construction of well-ordered upper and lower functions is too difficult (or even impossible).

References

[1] L. H. Erbe and Liu Xinzhi, Existence results for boundary value problems of second order impulsive differential equations, *J. Math. Anal. Appl.* **149** (1990), 56–69.

[2] I. Rachůnková and J. Tomeček, Impulsive BVPs with nonlinear boundary conditions for the second order differential equations without growth restrictions, *J. Math. Anal. Appl.*, to appear.

[3] I. Rachůnková and M. Tvrdý, Nonmonotone impulse effects in second order periodic boundary value problems, *Abstr. Anal. Appl.* **7** (2004), 577–590.

[4] I. Rachůnková and M. Tvrdý, Non–ordered lower and upper functions in second order impulsive periodic problems, *Dynamics of Contionuous, Discrete and Impulsive Systems*, to appear.

[5] I. Rachůnková and M. Tvrdý, Existence results for impulsive second order periodic problems, *Nonlinear Anal., Theory Methods Appl.*, **59** (2004), 133–146.

(Received October 7, 2004)

Department of Mathematics, Palacký University, Olomouc, Czech Republic
E-mail address: jan_tomecek@seznam.cz